NEWINGTON COLLEGE

Trial HSC Examination, 1991

12 MATHEMATICS

2/3 UNIT

Set by Mr O'Rourke

Time allowed - Three hours

(Includes reading time)

DIRECTIONS TO CANDIDATES:

- *All* questions may be attempted
- *All* questions are of equal value
- In every question, show all necessary working
- Write in *blue* or *black* and only on the lined side of the examination paper provided
- Marks may not be awarded for careless or badly arranged work.
- A table of standard integrals is provided for your convenience. Approved slide rules or silent calculators may be used.
- The answers to the ten questions in this paper are to be returned in separate bundles clearly marked Question 1, Question 2 etc.
- Each bundle must show your Candidate's number.
- The questions are not necessarily arranged in order of difficulty. Candidates are advised to read the whole paper carefully at the start of the examination.

QUESTION 1 *(12 Marks)*

(a) A man spent $\frac{1}{3}$ of his salary on clothes, $\frac{1}{4}$ on rent, $\frac{1}{5}$ on food and $\frac{1}{6}$ on entertainment. What percentage is left ?

(b) Solve $\frac{3-x}{4} - \frac{1-2x}{5} = 1$

(c) Find all possible values for x such that $|2x-3| = 6 - x$

(d) In ΔABC, the length of AB is 14.1 cm, CA 19.2 cm and $\angle BAC = 96°$.

(i) Use the cosine rule to find the length of BC, correct to 2 significant figures.

(ii) Calculate the area of ΔABC, correct to 3 significant figures.

QUESTION 2 *(12 Marks)*

(a) The point A(–4,–1) lies on the line *l*. The line *m*, whose equation is $2x + 3y = 2$, is perpendicular to *l*.

(i) Find the equation of *l*.

(ii) If *l* and *m* intersect at P, find the coordinates of P.

(iii) Find the distance from A to *m*.

(b) State, for the function $f(x) = \log_e x$.

(i) the natural (largest possible) domain and

(ii) the range

If $f(x) = 2.3$, find x correct to 3 significant figures.

QUESTION 3 *(12 Marks)*

(a) The first 3 terms of a Geometric series are 8, 4, 2.

(i) Show that T_n, the nth term of the series, is given by $T_n = 2^{4-n}$

(ii) If the last term of the series is $\frac{1}{128}$, how many terms are there in the series ?

(iii) Show that the sum of n terms is always less than 16.

(b) Given that $\sin(A+B) = \sin A\cos B + \cos A\sin B$, show that the exact value of

$\sin 75°$ is $\frac{\sqrt{3}+1}{2\sqrt{2}}$.

/ Q4 ...page 2

QUESTION 4 *(12 Marks)*

(a) Differentiate each of the following expressions with respect to x.

(i) xe^{3x}

(ii) $\dfrac{4}{x-2}$

(b) Write an indefinite integral for

(i) $\dfrac{2}{7x}$

(ii) $\dfrac{x-2}{\sqrt{x}}$

(c) If $\dfrac{dy}{dx} = \sec^2 2x$ and $y = 0$ when $x = \dfrac{\pi}{8}$, evaluate y when $x = \dfrac{\pi}{3}$.

QUESTION 5 *(12 Marks)*

(a) (i) Draw a neat sketch of the curve $y = x^2 - 4$, labelling its essential features.

(ii) The area between the curve $y = x^2 - 4$ and the x axis is rotated about the y axis. Find the volume of the solid so formed.

(b) A particle moves along the x axis with a displacement at time t seconds equal to $[\,t - 2\log_e(t+1)]$ metres.

(i) Find the initial velocity.

(ii) Show that the acceleration is positive for all values of t.

QUESTION 6 *(12 Marks)*

(a) (i) The end of a pendulum 1 metre long swings over an arc of length 16cm. Leaving your answer in terms of π, find the angle (in degrees) over which the pendulum swings.

(ii) The sector OAB represents the area swept out by the pendulum. Find this area

(b) Find the value of k for which the equation $x^2 - (k-1)x - 2(k-1) = 0$ has:

(i) a root equal to -1

(ii) equal roots

/ Q7...page 3

QUESTION 7 *(12 Marks)*

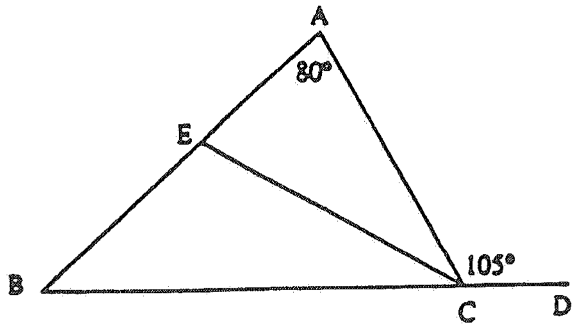
(a) In one box there two black balls and three red balls. In a second box there are three black balls and five green balls. If a ball is drawn from each box, what is the probability that

(i) both balls are black

(ii) one ball is red and the other is green

(iii) at least one ball is black ?

(b) (i) Redraw the diagram below on your examination paper and show all the given information.



Given: angle EAC = 80° , angle ACD = 105° , AE = AC.

(ii) Prove EB = EC

QUESTION 8 *(12 Marks)*

(a) For the curve $y = 9x - 3x^3$, find:

(i) the coordinates of its turning points

(ii) any points of inflexion

(iii) the domain for which the curve is concave down.

Sketch the curve labelling its essential features.

(b) A ship travelling at 10 metres per second) is subjected to water resistance proportional to the speed S. The engines are cut and the ship slows down according to the rule $\frac{dS}{dt} = -kS$, where t is the time in seconds and k is a constant. The speed after t seconds is given by $S = 10e^{-kt}$ metres per second. Find

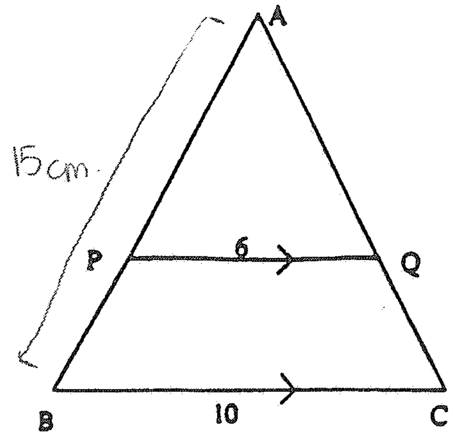
(i) k, if after 20 seconds S = 5 metres per second)

(ii) the speed after 1 minute.

/ Q9...page 4

QUESTION 9 *(12 Marks)*

(a)



Given: PQ is parallel to BC, AB = 15 cm

(i) Prove Δ's ABC and APQ are similar

(ii) Find AP

(b) Using the four digits 1, 2, 3, 4 all the possible two digit numbers are formed (i.e. repitition is allowed). If one of these two digit numbers is chosen at random, what is the probability that it is less than 32 ?

(c) If $a\sqrt{3} + 4\sqrt{b} = \sqrt{75} + \sqrt{80} + \sqrt{12}$, find a and b.

QUESTION 10 *(12 Marks)*

(a) Find $\lim_{x \to \infty} \dfrac{x - 1}{x^2 - 1}$

(b) Use the trapezoidal rule with two function values to find an approximation for,

$$\int_0^4 e^{x^2}\,dx$$

(c) A cylindrical can may be constucted in such a way that the sum of its height and its diameter will be 18 cm.

(i) Find the dimensions of the can that will make the volume a maximum.

(ii) Calculate this maximum volume, correct to the nearest cm^3

END OF PAPER

3 UNIT SOLUTIONS

1 (a) Fraction remaining

$= 1 - \left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}\right)$

$= \frac{1}{20}$

% remaining = 5 %

(b) $\frac{3-x}{4} - \frac{1-2x}{5} = 1$

$5(3-x) - 4(1-2x) = 20$

$15 - 5x - 4 + 8x = 20$

$3x = 9$

$x = 3$

(c) $2x - 3 = 6 - x$

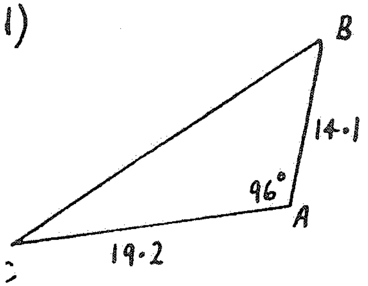
$x = 3$

or $2x - 3 = x - 6$

$x = -3$

solution: $x = \pm 3$

(d)



$BC^2 = 19.2^2 + 14.1^2 - 2 \times 14.1 \times 19.2 \cos 96°$

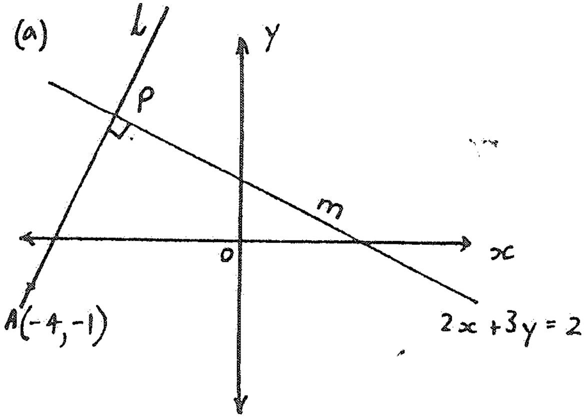
$= 624.0458912$ (Calculator Display)

BC = 25 cm (2 significant fig.)

(ii) Area $= \frac{1}{2} \times 19.2 \times 14.1 \sin 96°$

$= 135 \text{ cm}^2$ (3 significant fig)

2. (a)



(i) gradient of $m = -\frac{2}{3}$ ∴ gradient of $l = \frac{3}{2}$

required equation:-

$y + 1 = \frac{3}{2}(x + 4)$

$3x - 2y + 10 = 0$

(ii) $3x - 2y + 10 = 0$ ---- ①

$2x + 3y - 2 = 0$ — ②

Solve simultaneously

to get $x = -2, y = 2$

∴ $P(-2, 2)$

(iii) Distance $AP = \frac{|2 \times -4 + 3 \times -1 - 2|}{\sqrt{2^2 + 3^2}}$

$= \frac{13}{\sqrt{13}}$

$= \sqrt{13}$

(b)(i) Domain: $x > 1$

If $f(x) = 2.3$

$\log_e x = 2.3$

$x = e^{2.3}$

$= 9.97$ (correct to 3 signif. figures)

3. (a)(i) $a = 8,\ r = \frac{1}{2}$

$T_n = ar^{n-1}$

$= 8 . \left(\frac{1}{2}\right)^{n-1}$

$= 2^3 . (2^{-1})^{n-1}$

$= 2^3 . 2^{1-n}$

$= 2^{4-n}$

(ii) $2^{4-n} = \frac{1}{128} = 2^{-7}$

$4 - n = -7$

$n = 11$

(iii) $S_\infty = \frac{a}{1-r}$

$= \frac{8}{1 - \frac{1}{2}}$

$= 16$

∴ $S_n < 16$ for all values of n.

(b) $\sin(A+B) = \sin A \cos B + \cos A \sin B$

$\therefore \sin 75° = \sin(45° + 30°)$

$= \sin 45° \cos 30° + \cos 45° \sin 30°$

$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2}$

$= \frac{\sqrt{3} + 1}{2\sqrt{2}}$

4. (a)(i) $\frac{d}{dx}\left(xe^{3x}\right)$

$= x.3e^{3x} + 1.e^{3x}$

$= e^{3x}(3x+1)$

(ii) $\frac{d}{dx}\left(\frac{4}{x-2}\right)$

$= -4(x-2)^{-2}$

$= \frac{-4}{(x-2)^2}$

(b)(i) $\int \frac{2}{7x}\,dx = \frac{2}{7}\ln(x) + C$

(ii) $\int \frac{x-2}{\sqrt{x}}\,dx = \int \sqrt{x} - 2x^{-\frac{1}{2}}\,dx$

$= \frac{2}{3}x^{\frac{3}{2}} - 4x^{\frac{1}{2}} + C$

(c) $\frac{dy}{dx} = \sec^2 2x$

$y = \frac{1}{2}\tan 2x + C$, $\quad x = \frac{\pi}{8}, y = 0$

$0 = \frac{1}{2}.\tan\frac{\pi}{4} + C$

$\therefore C = -\frac{1}{2}$

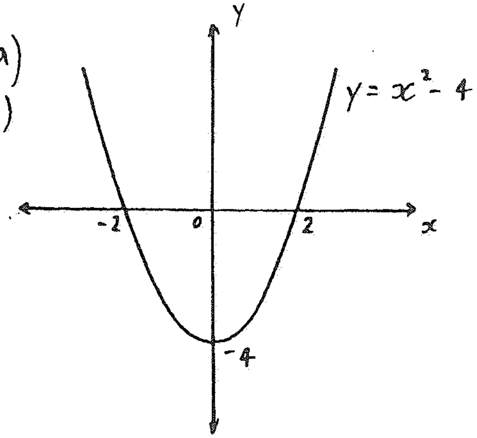
$\therefore y = \frac{1}{2}\tan 2x - \frac{1}{2}$, $\quad x = \frac{\pi}{3}$

$y = \frac{1}{2}.\tan\frac{2\pi}{3} - \frac{1}{2}$

$= -\frac{1}{2}\left(\sqrt{3}+1\right)$

5. (a)

(i)



(ii) Volume $= \pi\int_{-4}^{0} x^2\,dy$

$= \pi\int_{-4}^{0}(y+4)\,dy$

$= \pi\left[\frac{1}{2}y^2 + 4y\right]_{-4}^{0}$

$= \pi\left[0 - (8-16)\right]$

$= 8\pi$ units^3

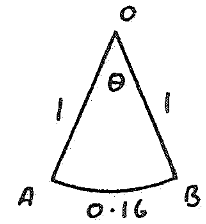
(b)(i) $v = \frac{dx}{dt} = 1 - \frac{2}{t+1}$ $\quad$ at $t = 0$

$= -1\ ms^{-1}$

(ii) acc$^n = \frac{dv}{dt} = \frac{2}{(t+1)^2}$

> 0 for all values of t.

6. (a)(i)



$0{\cdot}16 = 1 \times \theta$

$\therefore \theta = 0{\cdot}16^c$

$= \frac{0{\cdot}16 \times 180}{\pi}^{\circ}$

$= \frac{28{\cdot}8}{\pi}^{\circ}$

(ii) Area OAB $= \frac{1}{2}.1^2.0{\cdot}16$

$= 0{\cdot}08\ m^2$

(b)(i) $x^2 - (k-1)x - 2(k-1) = 0$

$x = -1$, $\quad (-1)^2 - (k-1).-1 - 2(k-1) = 0$

$1 + k - 1 - 2k + 2 = 0$

$k = 2$

(ii) Equal roots $\Rightarrow \Delta = 0$

$\Delta = (-(k-1))^2 - 4.1.-2(k-1)$

$\therefore k^2 + 6k - 7 = 0$

$(k+7)(k-1) = 0$

$k = -7$ or $k = 1$

7.

(a) 2B, 3R $\quad$ 3B, 5G

(i) $P(BB) = \frac{2}{5}\cdot\frac{3}{8}$

$= \frac{3}{20}$

(ii) $P(RG) = \frac{3}{5}\cdot\frac{5}{8}$

$= \frac{3}{}$

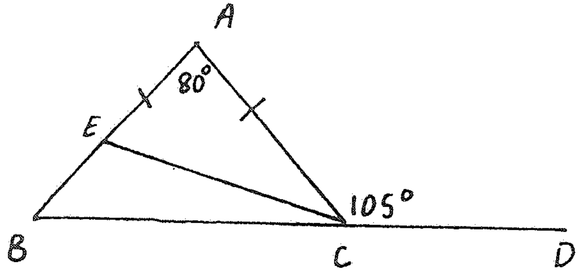
(ii) P(at least one black)

$= 1 - P(\text{No black})$

$= 1 - \frac{3}{5} \cdot \frac{5}{8}$

$= \frac{5}{8}$

(b)



$\angle ACD = \angle ABC + \angle CAB$ (exterior ∠ of ΔABC)

∴ $\angle ABC = 25°$

ΔAEC is isosceles (AE = AC)

∴ $\angle AEC = \angle ACE$ (base ∠'s of isosceles Δ)

$\angle EAC + \angle ACE + \angle CEA = 180°$ (∠ sum of Δ)

∴ $\angle AEC = 50°$

$\angle BEC = 180° - 50°$ (straight ∠)

$= 130°$

$\angle BEC + \angle EBC + \angle ECB = 180°$ (∠ sum of Δ)

∴ $\angle ECB = 25°$

∴ ΔEBC is isosceles ($\angle ECB = \angle EBC$)

∴ EB = EC

(i) $y = 9x - 3x^3$

$\frac{dy}{dx} = 9 - 9x^2$

$= 0$ when $x = \pm 1$

$x = 1,\ y = 6$

$x = -1,\ y = -6$

∴ turning points $(1, 6), (-1, -6)$

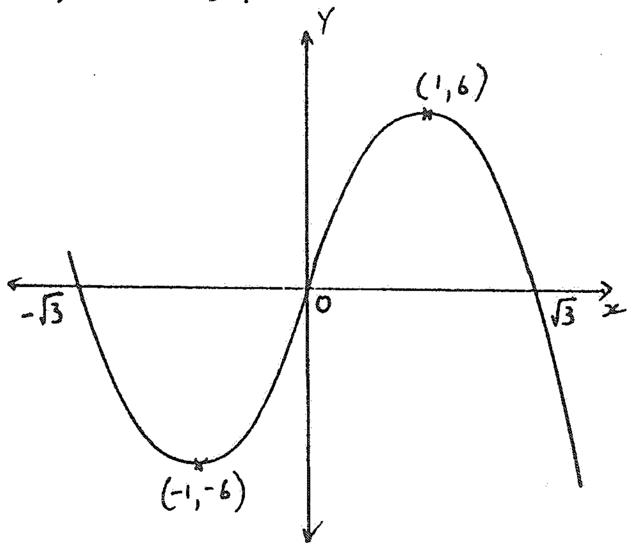
(ii) $\frac{d^2y}{dx^2} = -18x$

$= 0$ when $x = 0$

∴ pt of inflexion at $(0, 0)$

(iii) Concave down when $\frac{d^2y}{dx^2} < 0$

∴ $x > 0$.



(b) $S = 10e^{-kt}$

(i) $5 = 10e^{-20k}$

$-20k = \ln 0.5$

$k = -\frac{\ln 0.5}{20}$

(ii) $S = 10\, e^{+\frac{\ln 0.5}{20} \cdot 60}$

$= 10\, e^{\ln (0.5)^3}$

[illegible]

9. (a)(i) $\angle APQ = \angle ABC$ (Corresponding ∠'s in ∥ lines PQ, BC)

$\angle PAQ = \angle BAC$ (Common ∠'s)

$\angle AQP = \angle ACB$ (Remaining 3rd ∠'s must be equal)

∴ ΔAPQ ||| ΔABC

(ii) $\frac{AP}{AB} = \frac{PQ}{BC}$

$AP = \frac{15.6}{10}$

$= 9$ cm

(b)

	1	2	3	4
1	11	12	13	14
2	21	22	23	24
3	31	32	33	34
4	41	42	43	44

$N(S) = 16,\ N(E) = 9$

$P(\text{No.} < 32) = \frac{9}{16}$

(c) $a\sqrt{3} + 4\sqrt{b} = \sqrt{75} + \sqrt{80} + \sqrt{12}$

$= 5\sqrt{3} + 4\sqrt{5} + 2\sqrt{3}$

$= 7\sqrt{3} + 4\sqrt{5}$

∴ $a = 7,\ b = 5$

10/(a) $\lim_{x \to \infty} \frac{x-1}{x^2-1} = \lim_{x \to \infty} \frac{x-1}{(x-1)(x+1)}$

$= \lim_{x \to \infty} \frac{1}{x+1}$

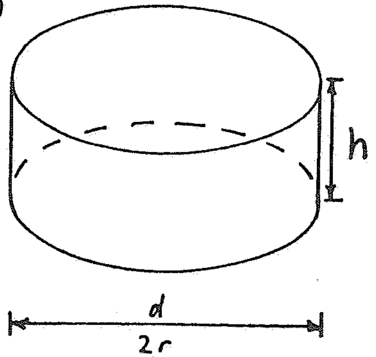
$= 0$

(b) $\int_0^4 e^{x^2} dx \doteqdot \frac{4}{2}\left[e^{0^2} + e^{4^2}\right]$

$= 2\left(1 + e^{16}\right)$

$\doteqdot 17772223$

(c)



(i) $h + 2r = 18$

$V = \pi r^2 h$

$= \pi r^2 (18 - 2r)$

$= 18\pi r^2 - 2\pi r^3$

$\frac{dV}{dr} = 36\pi r - 6\pi r^2$

$= 0$

$6\pi r(6 - r) = 0, \quad r = 0, 6$

$\frac{d^2V}{dr^2} = 36\pi - 12\pi r$

at $r = 6$, $\frac{d^2V}{dr^2} < 0$ $\therefore r = 6$ produces a max. V.

$\therefore$ dimensions:- diameter = 12 cm

height = 6 cm

(ii) Maximum Volume

$= \pi . 6^2 . 6$

$= 216\pi$

$= 679 \text{ cm}^3$ (to nearest cm^3)